Optimal Grading

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Introduction

From literature on subjective performance evaluation:

- the compression of ratings at job performance appraisals;

- the mismatch between students' grades and their abilities.

An agency problem with confronting interests: the teacher offers and commits to her designed grading standards (grades for performance/effort), upon observing which students choose what is best for them.

Key feature: *costless* rewards.

The Setup

$\approx$ a standard static principal-agent model with hidden information:

- the principal is a teacher, the agent - a student;
- the principal's goal to induce the agent to make a costly effort x (correlated with performance at the exam);
- the reward is a grade r: costless for the principal, but valuable to the agent;
- hidden information: student's ability $\theta \in [\theta_a, \theta_b]$ is his or her private information; and the teacher holds a belief $F(\theta)$;
- single-agent framework (for multiple agents, see, Dubey and Geanakoplos, 2005).

Optimization problem

The teacher is to find the effort-grade allocation $\{x(\theta), r(\theta)\}$ ($\equiv r(x)$) such that for every type θ and $\hat{\theta}$ it maximizes

$$\int_{\theta_a}^{\theta_b} U_P(x(\theta))dF(\theta)$$

s.t.

$$U_A(x(\theta), r(\theta), \theta) \geq U_A(x(\hat{\theta}), r(\hat{\theta}), \theta),$$

$$U_A(x(\theta), r(\theta), \theta) \geq 0,$$

$$0 \leq r(\theta) \leq 1.$$

Functional assumptions:

$$\begin{aligned} U_P(x) &= x; \quad U_A(x, r, \theta) = r - C(x, \theta); \\ C(x, \theta) &= g(x)t(\theta) = \frac{g(x)}{\theta}, \end{aligned}$$

with $C_x > 0, C_{xx} > 0, C_{x\theta} < 0$.

Solution

The intercomparison of the utilities is not possible.

Discretization: ability levels $\theta_1, \theta_2, ..., \theta_{n-1}, \theta_n$ with weights $p(\theta_i), i = 1, ..., n.$

Optimization problem

$$\sum_{i=1}^{n} p(\theta_i)x(\theta_i)$$

s.t.

$$r(\theta_i) - C(x(\theta_i), \theta_i) \geq 0,$$

$$r(\theta_i) - C(x(\theta_i), \theta_i) \geq r(\theta_j) - C(x(\theta_j), \theta_i),\ j \neq i,$$

$$0 \leq r(\theta_i) \leq 1.$$

Conjucture: the solution $\{(x(\theta_i), r(\theta_i)\}_{i=1}^{n}$ is a separating equilibrium.

The Lagrangian:

$$L(\{x(\theta_i)\}_{i=1}^n, \mu) = \sum_{i=1}^n p(\theta_i)x(\theta_i) +$$

$$+\mu(1 - \sum_{i=1}^n C(x(\theta_i), \theta_i) + \sum_{i=2}^n C(x(\theta_{i-1}), \theta_i)),$$

s.t.

$$x(\theta_i) \text{ is non-decreasing.}$$

The first-order conditions:

$$\frac{p(\theta_i)}{p(\theta_{i-1})} = \frac{C_x(x(\theta_i), \theta_i) - C_x(x(\theta_i), \theta_{i+1})}{C_x(x(\theta_{i-1}), \theta_{i-1}) - C_x(x(\theta_{i-1}), \theta_i)}, \quad i \neq n,$$

$$\frac{p(\theta_n)}{p(\theta_{n-1})} = \frac{C_x(x(\theta_n), \theta_n)}{C_x(x(\theta_{n-1}), \theta_{n-1}) - C_x(x(\theta_{n-1}), \theta_n)}.$$

$\equiv$ trade-offs between losses and gains

The right-hand side of the last FOC can be decomposed into

$$\frac{C_x(x(\theta_n), \theta_n)}{-\partial\theta(C_{x\theta}(x(\theta_n), \theta_n) - \partial x C_{x\theta x}(x(\theta_n), \theta_n))},$$

which is greater than 1 with $\partial\theta(= \frac{\theta_b - \theta_a}{n}) \rightarrow 0$.

$\longrightarrow$ No separating equilibrium.

Suppose $\{x(\theta_i), r(\theta_i)\}_{i=1}^{n}$ is a separating equilibrium.

Consider a change: $r(\theta_{n-1}) \longrightarrow r'(\theta_{n-1}) = 1$ and $x(\theta_{n-1}) \rightarrow x'(\theta_{n-1})$.

The loss: $[x(\theta_n) - x'(\theta_{n-1})]p(\theta_n)$

The gain: $[x'(\theta_{n-1}) - x(\theta_{n-1})]p(\theta_{n-1})$ plus the string of follow-up increases to the left from θ_{n-1}.

"Pooling at the top" condition:

$$\frac{P(\theta_m)}{p(\theta_{m-1})} = \frac{C_x(x(\theta_m), \theta_m)}{C_x(x(\theta_{m-1}), \theta_{m-1}) - C_x(x(\theta_{m-1}), \theta_m)}.$$

The continuous version is

$$\frac{1 - F(\theta^*)}{f(\theta^*)} = -\frac{C_x(x(\theta^*), \theta^*)}{C_{x\theta}(x(\theta^*), \theta^*)} = \theta^*$$

There is *a pooling equilibrium* for agent types θ in $[\theta^*, \theta_b]$:

$$\begin{aligned} r(\theta) &= 1, \\ x(\theta) &= x(\theta^*). \end{aligned}$$

The rest of the dynamics, for $\theta \in [\theta_a, \theta^*)$, is governed by

$$x(\theta) = g^{'-1}(A\theta^2 f(\theta)).$$

and

$$r(\theta) = \int_{\theta_a}^{\theta} A\tilde{\theta} f(\tilde{\theta}) x'(\tilde{\theta}) d\tilde{\theta} + B.$$

Result 1

- With a costless transfer structure in an agency problem, the "no distortion at the top" result ceases to hold.

$\longrightarrow$ *the compression of ratings*

Result 2

- The more pessimistic the principal is about her agents, the more generous she should be in motivating them.

More precisely, if

$$\frac{f_1(\theta)}{1 - F_1(\theta)} \leq \frac{f_2(\theta)}{1 - F_2(\theta)},$$

or

$$E_1(\theta \mid \theta > \bar{\theta}) \geq E_2(\theta \mid \theta > \bar{\theta}),$$

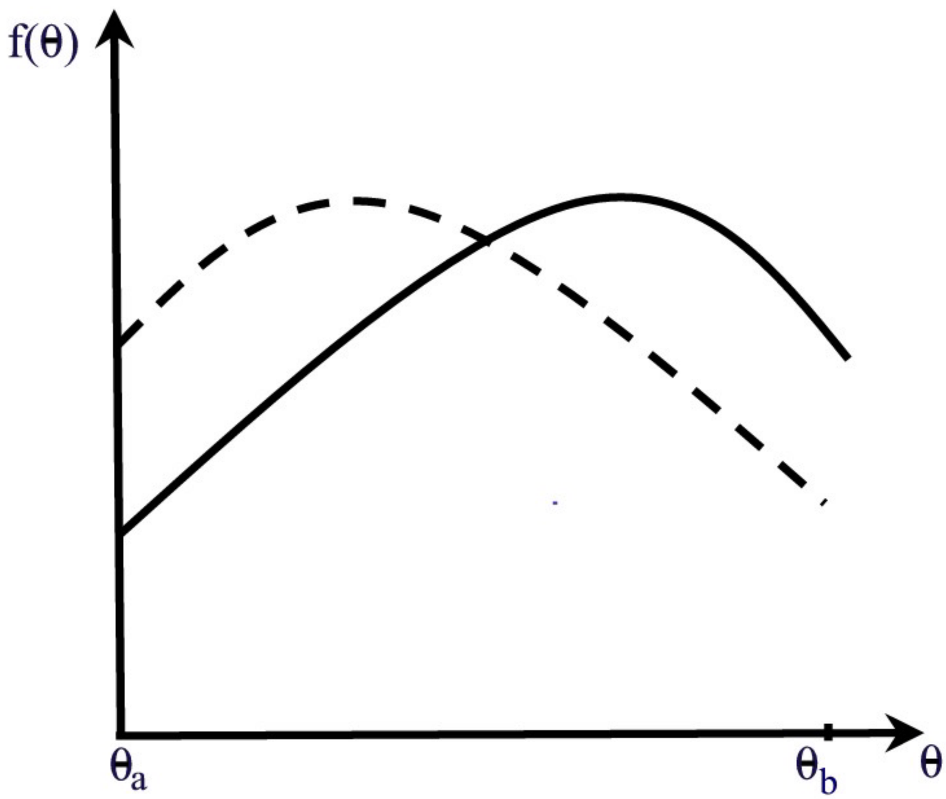
then

$$r_1(\theta) \leq r_2(\theta).$$

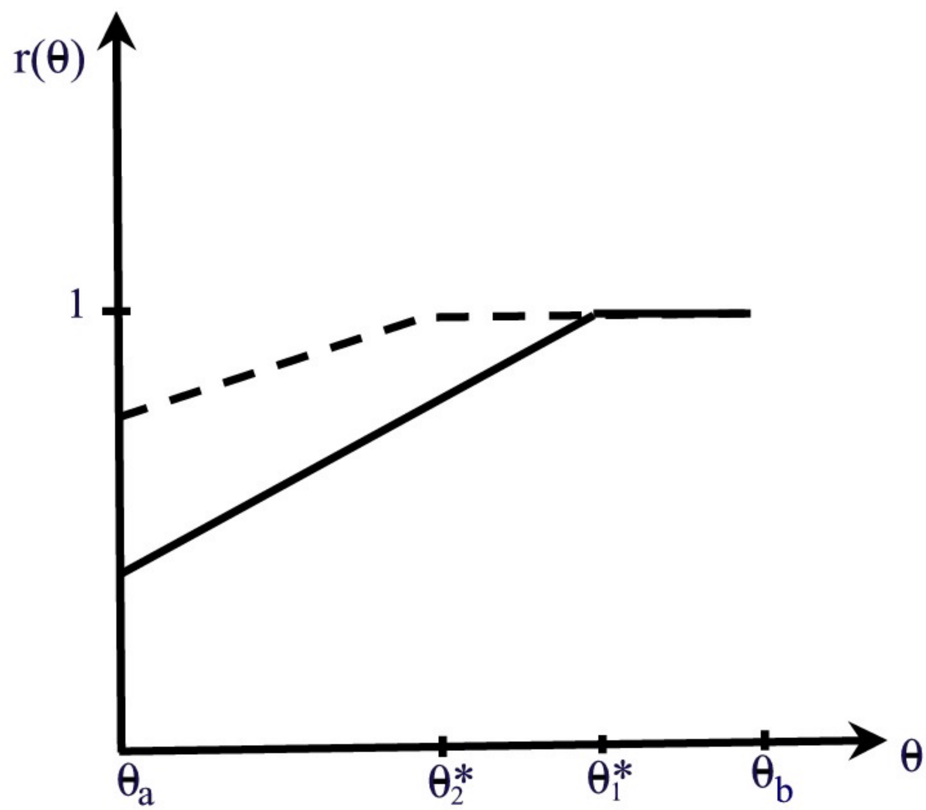
$\longrightarrow$ *The mismatch between students' grades and abilities*

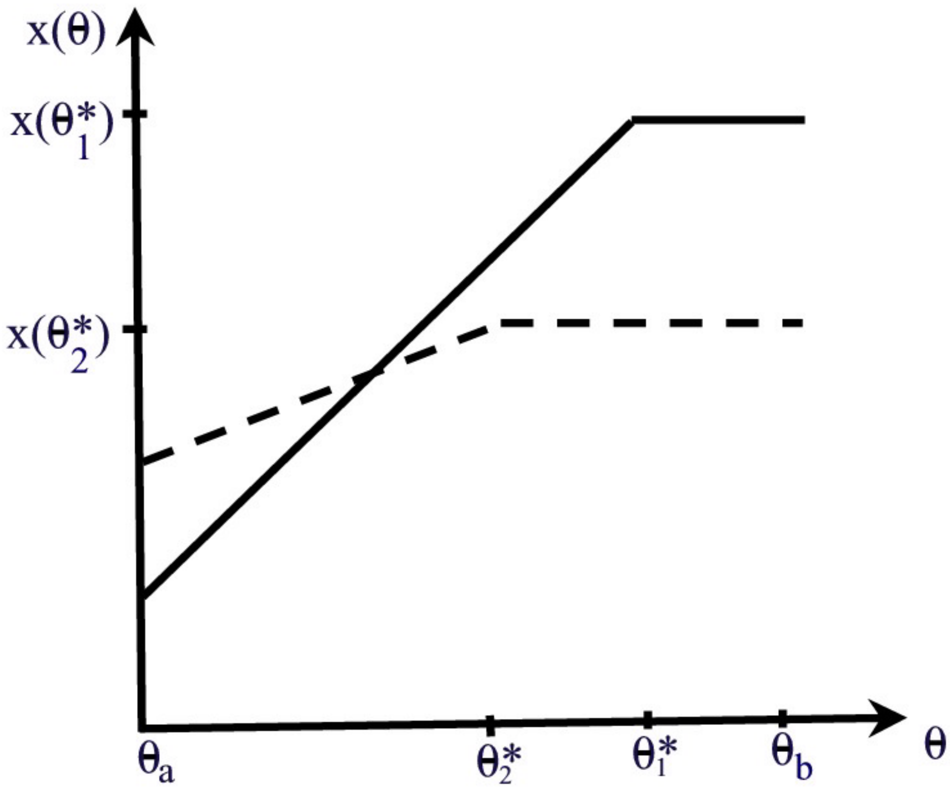
Figures

Distributions for abilities in two classes (solid graph for a more able class)



Optimal grade-effort allocations (solid line for a more able class)





x(θ)
x(θ*1)
x(θ*2)
θa
θ*2
θ*1
θb
θ

Empirical Evidence

Journal of Educational Measurement and *Educational and Psychological Measurement.*

The universally observed empirical fact is:

> *Studied fields with lower ability students as compared with those with higher ability students employ less stringent grading criteria.*

See, Aiken (1963), Goldman and Widawski (1976), Strenta and Elliot (1987), or Johnson (2003).

Concluding Remarks

- "Microfoundations" of grading;

- Normative and positive analysis of the introduction of incentive schemes for teachers;

- Intercomparison of grades between various departments: grade adjustment mechanisms.